

$A = f(C)$. For our purposes we replace condition (†) by the less restricting but closely related condition

$$(\dagger\dagger) \quad \limsup [f_n(C)] \subset_s f(C),$$

where $f: X \rightarrow Y$ is a mapping related in some specified way to the sequence of mappings $f_n: X \rightarrow Y$. We may then state the

THEOREM. *Let $f_n: X \rightarrow Y$ be a sequence of quasi-open mappings and suppose there exists a mapping $f: X \rightarrow Y$ with point inverses having compact components such that for each $x \in X$ there exists an arbitrarily close region R in X with boundary C about the component of $f^{-1}f(x)$ containing x such that (††) holds on C . Then $f_n(x)$ converges almost uniformly to $f(x)$ on X .*

As already indicated, this theorem has implications for sequences of functions similar to those mentioned in the earlier sections for the corresponding theorems on sequences of monotone mappings. The application of the theorem to the case of a closed algebra of mappings of X into Y is of special interest and will receive attention in a later paper.

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¹ G. T. Whyburn, "Regular convergence and monotone transformations," *Am. J. Math.*, **57**, 902-906, 1935.

² K. Menger, *Kurventheorie* (Berlin: B. G. Teubner, 1932).

³ L. Whyburn, *Ergeb. Math. Kolloq.* **2**, p. 11, 1930. This theorem may be found also in Menger's *Kurventheorie*, pp. 278-279.

SOLUTION OF THE BOLTZMANN-HILBERT INTEGRAL EQUATION II. THE COEFFICIENTS OF VISCOSITY AND HEAT CONDUCTION

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1. *Introduction.*—In this investigation we apply the method developed in Part I¹ to the determination of the coefficients of viscosity and heat conduction and of the perturbations in the respective distribution functions in a simple gas of the rigid sphere model. The essence of the method consists in *reducing the Boltzmann-Hilbert integral equation*, which controls the distribution function $f(\mathbf{c}, \mathbf{r}, t)$ of the molecules in space \mathbf{r} and in velocity-space \mathbf{c} , *to an ordinary differential equation*. In the cases of viscosity and heat conduction, the respective differential equations are of the fourth order. These are integrated numerically on the electronic computer (WEIZAC) to obtain the distribution functions, and from the latter the coefficients of viscosity and heat conduction are evaluated.

Let the gas molecules be rigid spheres of diameter σ and mass m . Further, let

$$f^0 = n \left(\frac{m}{2\pi kT} \right)^{3/2} \exp \left(- \frac{mC^2}{2kT} \right), \quad \mathbf{C} = \mathbf{c} - \mathbf{c}_0,$$

$$f = f^0(1 + \varphi), \quad \mathbf{p} = \mathbf{C} \sqrt{\frac{m}{2kT}}, \quad (1)$$

$$\mathfrak{D}f = \frac{\partial f}{\partial t} + \mathbf{c} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{c}}, \quad (2)$$

where T denotes the temperature of the gas, n the number density, k Boltzmann's constant, and \mathbf{F} the force per unit mass acting on the molecules. Then on treating φ as a small quantity of the first order, the Boltzmann-Hilbert integral equation takes on the form

$$\mathfrak{D}f^0 = -\frac{\sigma^2 n^2 m}{2\pi kT} \left\{ M(p) e^{-2p^2} \varphi(\mathbf{p}) + \frac{1}{\pi e} e^{-p^2} \int \varphi(\mathbf{p}_1) e^{-p_1^2} \left(R - \frac{2}{R} e^{\omega^2} \right) d\mathbf{p}_1 \right\} \quad (3)$$

where

$$M(p) = 1 + \left(2p + \frac{1}{p} \right) P(p), \quad P(p) = e^{p^2} \int_0^p e^{-x^2} dx, \quad (4)$$

$$R = |\mathbf{p} - \mathbf{p}_1|, \quad \omega = \frac{p p_1 \sin \theta'}{R}, \quad (5)$$

and θ' denotes the angle between \mathbf{p} and \mathbf{p}_1 .

In evaluating $\mathfrak{D}f^0$, we allow for variation in space of the quantities n , \mathbf{C} , and T , obtaining

$$\mathfrak{D}f^0 = f^0 \left\{ \frac{1}{n} \mathbf{C} \cdot \frac{\partial n}{\partial \mathbf{r}} + \left(p^2 - \frac{5}{2} \right) \mathbf{C} \cdot \frac{\partial \ln T}{\partial \mathbf{r}} + 2 \left(p_i p_k - \frac{1}{3} p^2 \delta_{ik} \right) \frac{\partial u_{0i}}{\partial x_k} \right\}. \quad (6)$$

Here p_i denotes the components of \mathbf{p} , and u_{0i} the components of \mathbf{c}_0 . The first term in equation (6) is proportional to the density gradient and enters into the determination of the coefficient of self-diffusion, which was treated in Part I. In this investigation we shall treat the problems of heat conduction and viscosity, for which $\mathfrak{D}f^0$ is represented by the second and third terms, respectively, in equation (6).

2. *The Differential Equation for the Distribution Function in the Case of Viscosity.*—In treating viscosity, we let n and T be constant, so that $\mathfrak{D}f^0$ reduces to the terms proportional to the velocity gradients in equation (6). An appropriate form for φ in this case is

$$\varphi = -\frac{1}{n\sigma^2} \sqrt{\frac{2m}{\pi kT}} b(p) \left(p_i p_k - \frac{1}{3} p^2 \delta_{ik} \right) \frac{\partial u_{0i}}{\partial x_k}, \quad (7)$$

where the summation convention is adhered to. On substituting equations (6) and (7) in equation (3) and equating the coefficients of $\partial u_{0i}/\partial x_k$, we get

$$p_i p_k - \frac{1}{3} p^2 \delta_{ik} = \left[\left(p_i p_k - \frac{1}{3} p^2 \delta_{ik} \right) M(p) e^{-p^2} b(p) + \frac{1}{\pi} \int \left(p_{1i} p_{1k} - \frac{1}{3} p_1^2 \delta_{ik} \right) b(p_1) e^{-p_1^2} \left(R - \frac{2}{R} e^{\omega^2} \right) d\mathbf{p}_1 \right]. \quad (8)$$

It is convenient to let $i = k = 3$ in equation (8), whereby it takes on the form

$$p^2 P_2(\cos \theta) = \left[p^2 M(p) e^{-p^2} b(p) P_2(\cos \theta) + \frac{1}{\pi} \int p_1^2 P_2(\cos \theta_1) b(p_1) e^{-p_1^2} \left(R - \frac{2}{R} e^{\omega^2} \right) d\mathbf{p}_1 \right]. \quad (9)$$

As was pointed out in Part I, this integral equation involves only the component $A_2(p, p_1)$ of the expansion of the kernel $[R - (2/R)e^{\omega^2}]$ into spherical harmonics of the angle θ' between \mathbf{p} and \mathbf{p}_1 :

$$\left[R - \left(\frac{2}{R} \right) e^{\omega^2} \right] = \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) A_n(p, p_1) P_n(\cos \theta'). \quad (10)$$

Using equation (10) in equation (9), we obtain

$$\begin{aligned} \int p_1^2 e^{-p_1^2} b(p_1) \left(R - \frac{2}{R} e^{\omega^2} \right) P_2(\cos \theta_1) d\mathbf{p}_1 \\ = 2\pi P_2(\cos \theta) \int_0^{\infty} p_1^4 e^{-p_1^2} b(p_1) A_2(p, p_1) dp_1, \end{aligned} \quad (11)$$

whereby the three-dimensional integral equation (9) reduces to the following one-dimensional integral equation:

$$p^2 = p^2 M(p) e^{-p^2} b(p) + 2 \int_0^{\infty} e^{-p_1^2} p_1^4 A_2(p, p_1) b(p_1) dp_1. \quad (12)$$

The kernel $A_2(p, p_1)$, which is symmetrical in the arguments p and p_1 , was previously² evaluated in the explicit form

$$\begin{aligned} p^3 p_1^3 A_2(p, p_1) &= \left[\frac{2}{35} p_1^7 - 3p_1^3 + 18p_1 + (-6p_1^4 + 15p_1^2 - 18)P(p_1) \right] \\ &+ p^2 \left[-\frac{2}{15} p_1^5 + 3p_1 + (2p_1^2 - 3)P(p_1) \right], \quad p_1 < p, \end{aligned} \quad (13)$$

$$\begin{aligned} &= \left[\frac{2}{35} p^7 - 3p^3 + 18p + (-6p^4 + 15p^2 - 18)P(p) \right] \\ &+ p_1^2 \left[-\frac{2}{15} p^5 + 3p + (2p^2 - 3)P(p) \right], \quad p_1 > p. \end{aligned} \quad (14)$$

Putting

$$\nu(p) \equiv p^5 - p^5 M(p) e^{-p^2} b(p), \quad (15)$$

the integral equation (12) takes on the form

$$\begin{aligned} \nu(p) &= \int_0^p e^{-x^2} b(x) \left[\frac{4}{35} x^8 - 6x^4 + 36x^2 + (-12x^5 + 30x^3 - 36x)P(x) \right] dx \\ &+ p^2 \int_0^p e^{-x^2} b(x) \left[-\frac{4}{15} x^6 + 6x^2 + (4x^3 - 6x)P(x) \right] dx \end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{4}{15}p^5 + 6p + (4p^2 - 6)P(p) \right] \int_p^\infty e^{-x^2} b(x) x^3 dx \\
& + \left[\frac{4}{35}p^7 - 6p^3 + 36p + (-12p^4 + 30p^2 - 36)P(p) \right] \int_p^\infty e^{-x^2} b(x) x dx.
\end{aligned} \tag{16}$$

We now proceed to reduce this integral equation to a differential equation. We have

$$\begin{aligned}
\frac{1}{p} \frac{dv}{dp} &= \int_0^p e^{-x^2} b(x) \left[-\frac{8}{15}x^6 + 12x^2 + (8x^3 - 12x)P(x) \right] dx \\
&+ \left[-\frac{4}{3}p^3 + 4p + (8p^2 - 4)P(p) \right] \int_p^\infty e^{-x^2} x^3 b(x) dx \\
&+ \left[\frac{4}{5}p^5 - 12p^3 + 12p + (-24p^4 + 12p^2 - 12)P(p) \right] \int_p^\infty e^{-x^2} b(x) x dx.
\end{aligned} \tag{17}$$

Let

$$y(p) = -\frac{1}{2}p^2 \int_p^\infty e^{-x^2} b(x) x dx + \frac{1}{2} \int_p^\infty e^{-x^2} b(x) x^3 dx, \tag{18}$$

so that

$$\int_p^\infty e^{-x^2} b(x) x dx = -\frac{y}{p}, \tag{19}$$

$$\int_p^\infty e^{-x^2} b(x) x^3 dx = 2y - p\dot{y}, \quad e^{-p^2} p^3 b(p) = p\ddot{y} - \dot{y}, \tag{20}$$

the dots denoting differentiation with respect to p .

Using these substitutions in equation (17), we are left with only the first integral, and the latter can be reduced by an additional differentiation, yielding the following differential equation for the determination of $y(p)$:

$$\begin{aligned}
& [p^2 + (p + 2p^3)P(p)]y^{IV} + [6p + 4p^3 + (2 + 16p^2 + 8p^4)P(p)]\ddot{y} \\
& + \left[1 + 20p^2 + 4p^4 + \left(-\frac{1}{p} + 10p + 44p^3 + 8p^5 \right) P(p) \right] \ddot{y} \\
& + \left[-\frac{1}{p} + 4p + 12p^3 + \left(\frac{1}{p^2} - 10 + 20p^2 + 24p^4 \right) P(p) \right] \dot{y} \\
& + [8p^2 + (16p + 32p^3)P(p)]y = 15p^2.
\end{aligned} \tag{21}$$

We are interested in a solution of equation (21) which is regular in the whole range of p from 0 to ∞ .

A result similar to, but not identical with, equation (21) was obtained by Boltzmann³ toward the end of his third paper on viscosity. Boltzmann derived an equation⁴ which is equivalent to equation (17) above; however, in deducing from it the fourth-order differential equation for y , he committed several errors. His equation (38) should read

$$\begin{aligned}
 15x = [2x^3 + (x^2 + 2x^3)\xi]Y^{IV} + [12x^2 + 4x^3 + (4x + 14x^2 + 4x^3)\xi]\ddot{Y} \\
 + [11x + 16x^2 + 2x^3 + (2 + 16x + 17x^2 + 2x^3)\xi]\dot{Y} + [6x + 4x^2 \\
 + (8x + 4x^2)\xi]\dot{Y} + [x + (1 + 2x)\xi]Y,
 \end{aligned} \quad (22)$$

where

$$x = p^2, \quad \xi = 2pP(p), \quad Y(x) = 8y(p). \quad (23)$$

Equation (22) is identical with equation (21).

The method by which Boltzmann arrived at the differential equation (22) is extremely laborious, the derivation covering 160 pages, of which over 40 are pure mathematics,⁵ in the sense that they are either entirely word-pure or are adorned with just one sentence.

3. *The Differential Equation for the Distribution Function in the Case of Heat Conduction.*—With n and c_0 constant in space, equation (6) reduces to

$$\mathfrak{D}f^0 = f^0 \left(p^2 - \frac{5}{2} \right) C \cdot \frac{\partial \ln T}{\partial r} = \sqrt{\frac{2kT}{m}} f^0 \left(p^2 - \frac{5}{2} \right) p \cdot \frac{\partial \ln T}{\partial r}. \quad (24)$$

A suitable form for φ now is

$$\varphi = - \frac{1}{n\sigma^2\sqrt{\pi}} a(p)p \cdot \frac{\partial \ln T}{\partial r}. \quad (25)$$

When equations (24) and (25) are substituted in equation (3) and the coefficients of $\partial T/\partial r$ are equated, we get the following integral equation for the determination of $a(p)$:

$$M(p)e^{-p^2} a(p)p + \frac{1}{\pi} \int a(p_1)p_1 e^{-p_1^2} \left(R - \frac{2}{R} e^{\omega^2} \right) dp_1 = \left(p^2 - \frac{5}{2} \right) p. \quad (26)$$

By now taking p in the direction of z , equation (26) assumes the form

$$\begin{aligned}
 pM(p)e^{-p^2} a(p)P_1(\cos \theta) + \frac{1}{\pi} \int p_1 a(p_1)e^{-p_1^2} P_1(\cos \theta_1) \left(R - \frac{2}{R} e^{\omega^2} \right) dp_1 \\
 = \left(p^2 - \frac{5}{2} p \right) P_1(\cos \theta). \quad (27)
 \end{aligned}$$

This equation evidently involves only the first harmonic $A_1(p, p_1)$ in the expansion (10) of the kernel $[R - (2/R)e^{\omega^2}]$:

$$\begin{aligned}
 \int p_1 a(p_1)e^{-p_1^2} P_1(\cos \theta_1) \left(R - \frac{2}{R} e^{\omega^2} \right) dp_1 \\
 = 2\pi P_1(\cos \theta) \int_0^\infty p_1^3 a(p_1)e^{-p_1^2} A_1(p, p_1) dp_1, \quad (28)
 \end{aligned}$$

so that equation (27) reduces to

$$pM(p)e^{-p^2} a(p) + 2 \int_0^\infty e^{-p_1^2} p_1^3 a(p_1) A_1(p, p_1) dp_1 = p^2 - \frac{5}{2} p. \quad (29)$$

Here²

$$p^2 p_1^2 A_1(p, p_1) = \frac{2}{15} p_1^5 - 4g(p_1) - \frac{2}{3} p_1^3 p^2, \quad p_1 < p, \quad (30)$$

$$= \frac{2}{15} p^5 - 4g(p) - \frac{2}{3} p^3 p_1^2, \quad p_1 > p, \quad (31)$$

where

$$g(p) = [p + (p^2 - 1)P(p)], \quad P(p) = e^{p^2} \int_0^p e^{-x^2} dx. \quad (32)$$

Thus $a(p)$ is a solution of the one-dimensional integral equation

$$\begin{aligned} \gamma(p) &\equiv p^5 - \frac{5}{2}p^3 - p^3 M(p) e^{-p^2} a(p) \\ &= \int_0^p e^{-x^2} a(x) x \left[\frac{4}{15} x^5 - 8g(x) - \frac{4}{3} p^2 x^3 \right] dx \\ &\quad + \int_p^\infty e^{-x^2} a(x) x \left[\frac{4}{15} p^5 - 8g(p) - \frac{4}{3} p^3 x^2 \right] dx. \end{aligned} \quad (33)$$

As in the previous section, we proceed to turn the integral equation (33) into a differential equation by first introducing an auxiliary function,

$$S(p) = - \int_p^\infty e^{-x^2} x a(x) dx. \quad (34)$$

Repeated differentiation of equation (33) yields

$$\frac{d^2}{dp^2} \left(\frac{1}{p} \frac{d\gamma}{dp} \right) = 8 \frac{d^2}{dp^2} \left(\frac{S}{p} \frac{dg}{dp} \right) - 8pS, \quad (35)$$

or

$$\begin{aligned} [p^2 + (2p^3 + p)P(p)]S^{IV} &+ [7p + 6p^3 + (1 + 20p^2 + 12p^4)P(p)]\ddot{S} \\ &+ \left[2 + 42p^2 + 12p^4 + \left(-\frac{2}{p} + 26p + 96p^3 + 24p^5 \right) P(p) \right] \ddot{S} \\ &+ \left[-\frac{2}{p} + 38p + 72p^3 + 8p^5 + \left(\frac{2}{p^2} - 2 + 140p^2 + 152p^4 + 16p^6 \right) P(p) \right] \dot{S} \\ &+ [56p^2 + 32p^4 + (32p + 160p^3 + 64p^5)P(p)]S = 30p^2. \end{aligned} \quad (36)$$

The solution $S_1(p)$ of equation (36), which is required for our application, is to be regular in the whole range of p extending from 0 to ∞ . In addition to $S_1(p)$, the homogeneous differential equation (36) also possesses the regular solution

$$S_0(p) = Ae^{-p^2}, \quad (37)$$

so that

$$S(p) = S_1(p) + Ae^{-p^2}. \quad (38)$$

The availability of $S_0(p)$ enables us to satisfy a condition on $S(p)$:

$$\int_0^\infty S(p)p^2 dp = 0, \quad (39)$$

which is required⁶ in order that

$$\int f^0 m C dc = \int f m C dc. \quad (40)$$

Equation (40) expresses the normalization condition that the mean flow be given by c_0 , even in the presence of the temperature gradient.

4. *The Coefficients of Heat Conduction and Viscosity.*—The flow of heat, q , is given by

$$\begin{aligned} q &= \frac{m}{2} \int f^0 \varphi C^2 C dc = -\frac{m}{2\pi^2 \sigma^2} \left(\frac{2kT}{m} \right)^{1/2} \int e^{-p^2} p^2 \left(\mathbf{p} \cdot \frac{\partial \ln T}{\partial \mathbf{r}} \right) \mathbf{p} a(p) d\mathbf{p} \\ &= -\frac{4k}{3\pi \sigma^2} \sqrt{\frac{2kT}{m}} \frac{\partial T}{\partial r} \int_0^\infty e^{-p^2} a(p) p^4 dp = -\lambda \frac{\partial T}{\partial r}. \end{aligned} \quad (41)$$

Hence we get for the coefficient of heat conduction, λ ,

$$\lambda = \frac{4k}{3\pi \sigma^2} \sqrt{\frac{2kT}{m}} \int_0^\infty e^{-p^2} a(p) p^4 dp. \quad (42)$$

In the case of viscosity, the non-hydrostatic part of the stress tensor, τ_{ji}^1 is expressed by

$$\begin{aligned} \tau_{ji}^1 &= m \int f^0 \varphi C_j C_i dc = \\ &= -\frac{2kT}{\pi^2 \sigma^2} \sqrt{\frac{2m}{kT}} \int e^{-p^2} b(p) \left(p_i p_j - \frac{1}{3} p^2 \delta_{ij} \right) \frac{\partial u_{0i}}{\partial x_k} p_j p_i d\mathbf{p}. \end{aligned} \quad (43)$$

When this is evaluated, we obtain expressions for τ_{ji}^1 of the Stokes-Navier form, with a coefficient of viscosity, μ , given by

$$\mu = \frac{8\sqrt{2mkT}}{15\pi \sigma^2} \int_0^\infty e^{-p^2} b(p) p^6 dp. \quad (44)$$

5. *Integration of the Differential Equations.*—Taking first the differential equation (36) for the distribution function $S(p)$ in the case of heat conduction, we find from the indicial equation at the origin the following allowed values for the leading powers n of p :

$$n = -1, 0, 1, 2. \quad (45)$$

Of these, the value 1 is disallowed, because, according to equation (34), $\dot{S}(0)$ must vanish; and so is also the value $n = -1$. The index $n = 0$ gives the solution (37), and there is another regular solution, $S_2(p)$, starting with p^2 . The inhomogeneous solution $S_4(p)$ of equation (36) starts with the power p^4 .

At infinity, there are, in addition to equation (37), four solutions with the following leading terms in their asymptotic expansions:

$$pe^{-p^2}, \quad \frac{1}{p} e^{-p^2}, \quad \frac{1}{p^2} e^{-p^2}, \quad \frac{1}{p^4}, \quad (46)$$

of which the first is the inhomogeneous solution. Both $S_2(p)$ and $S_4(p)$ were found upon numerical integration to approach the $(1/p^4)$ -solution for large values of p . The ratio of $S_2(p)$ to $S_4(p)$ therefore approaches a constant, whereby one can determine a linear combination of $S_2(p)$ and $S_4(p)$ which is free from the $(1/p^4)$ -solution. This procedure was found to be sufficiently accurate to obviate the need of integrating the first three solutions in (46) backward from large values of p , and then matching them at some intermediate value with a linear combination of $S_2(p)$ and $S_4(p)$.

The solution $S(p)$ thus obtained, regular in the whole range of p , is shown in Table 1. Expression (42) for the coefficient of heat conduction, λ , can be written in the non-dimensional form,

$$\frac{\lambda}{\lambda_1} = -\frac{256}{45} \sqrt{\frac{2}{\pi}} \int_0^\infty p^4 S(p) dp, \quad (47)$$

where

$$\lambda_1 = \frac{75k}{64\sigma^2} \sqrt{\frac{kT}{\pi m}}. \quad (48)$$

Using the values of $S(p)$ obtained by the numerical integration of equation (36), we obtain a value of 1.025218 for (λ/λ_1) . The values for (λ/λ_1) obtained in the successive approximations of the Chapman-Enskog method⁷ are

$$1, \quad 1.02273, \quad 1.02482, \quad 1.02513. \quad (49)$$

In the case of the distribution function $y(p)$ for viscosity, which is governed by the differential equation (21), we find, from the indicial equation at the origin, the following allowed values for the leading powers n of p :

$$n = -1, 0, 1, 2, \quad (50)$$

of which, in addition to $n = -1$, the value $n = 1$ is also disallowed on account of the requirement, resulting from equation (19), that $\dot{y}(0)$ must vanish. Near the origin we need to integrate the solutions $y_0(p)$ and $y_2(p)$, belonging to $n = 0$ and $n = 2$, respectively, as well as the inhomogeneous solution $y_4(p)$ starting with p^4 .

For large values of p , the leading terms in the asymptotic solutions of equation (21) are

$$p^{-1}e^{-p^2}, \quad p^{-3-i\sqrt{3}}e^{-p^2}, \quad p^{-3+i\sqrt{3}}e^{-p^2}, \quad p^{-1-i\sqrt{3}}, \quad p^{-1+i\sqrt{3}}, \quad (51)$$

of which the first is the inhomogeneous solution. Asymptotically, the solutions $y_0(p)$, $y_2(p)$, and $y_4(p)$ are dominated by the last two solutions of (51). As in the case of heat conduction, these solutions were eliminated by using the limiting values of the cross-ratios of $y_0(p)$, $y_2(p)$, and $y_4(p)$ for large p . The resulting solution, $y(p)$, thus obtained, regular in the whole range of p , is shown in Table 1.

The expression for the coefficient of viscosity μ given in equation (44) can be put in the form

$$\frac{\mu}{\mu_1} = \frac{256}{5\sqrt{2}\pi} \int_0^\infty p^2 y(p) dp, \quad (52)$$

with

$$\mu_1 = \frac{5\sqrt{kmT}}{16\sqrt{\pi\sigma^2}}. \quad (53)$$

Values for (μ/μ_1) obtained in the successive approximations of the Chapman-Enskog method⁷ are

$$1, \quad 1.01485, \quad 1.01588, \quad 1.01600. \quad (54)$$

Using the values of $y(p)$ obtained by numerical integration of equation (21), we obtain the value of 1.016034 for (μ/μ_1) .

In the first-order theory presented in this paper it was assumed that φ in equation (1) is less than 1. We see from Table 1 that the quantities $p^2b(p)$ and $pa(p)$, which

TABLE 1*

$$p = C \sqrt{\frac{m}{k_2 T}}$$

p	$S(p)$	$pa(p)$	$y(p)$	$p^2b(p)$
0	0.597142	0	0.125867	0
0.1	.586378	-0.215765	.124491	0.006196
0.2	.555074	-0.424299	.120458	0.024668
0.3	.506062	-0.618610	.114028	0.055080
0.4	.443610	-0.792152	.105611	0.096902
0.5	.372844	-0.938986	.095711	0.149447
0.6	.299102	-1.053877	.084882	0.211920
0.7	.227314	-1.132324	.073677	0.283462
0.8	.161529	-1.170543	.062599	0.363191
0.9	.104613	-1.165432	.052070	0.450239
1.0	.058160	-1.114455	.042407	0.543772
1.1	.022569	-1.015593	.033820	0.643015
1.2	-.002738	-0.867241	.026416	0.747251
1.3	-.019035	-0.668134	.020210	0.855835
1.4	-.027987	-0.417277	.015146	0.968190
1.5	-.031383	-0.113887	.011121	1.083802
1.6	-.030923	0.242651	.008000	1.202218
1.7	-.028074	.652825	.005639	1.323043
1.8	-.024002	1.117023	.003895	1.445929
1.9	-.019556	1.635558	.002637	1.570574
2.0	-.015295	2.208681	.001749	1.696714
2.1	-.011537	2.836601	.001138	1.824122
2.2	-.008421	3.519490	.000725	1.952596
2.3	-.005962	4.257492	.000453	2.081966
2.4	-.004101	5.050729	.000277	2.212081
2.5	-.002744	5.899307	.000166	2.342811
2.6	-.001789	6.803316	.000098	2.474043
2.7	-.001136	7.762836	.000057	2.605678
2.8	-.000704	8.777936	.000032	2.737632
2.9	-.000426	9.848679	.000018	2.869832
3.0	-.000252	10.975	.000010	3.0022
3.1	-.000145	12.157	.000005	3.1347
3.2	-.000082	13.395	.000003	3.2673
3.3	-.000045	14.689	.000001	3.3999
3.4	-.000024	16.039	0.000001	3.5325
3.5	-.000013	17.444	...	3.6651
3.6	-.000007	18.906	...	3.7977
3.7	-.000003	20.423	...	3.9301
3.8	-.000002	21.997	...	4.0625
3.9	-0.000001	23.626	...	4.1947
4.0	...	25.319	...	4.3268
5.0	...	45	...	5.6

* The dependence of the perturbation in the distribution function on p is given by $pa(p)e^{-p^2}$ in the case of heat conduction and by $p^2b(p)e^{-p^2}$ in the case of viscosity.

represent the factors depending on p in equations (7) and (25), become large for large values of p . Hence the linearized approximation becomes relatively poorer for large values of momenta.

Summary.—In a previous publication¹ it was shown that it is possible to reduce the Boltzmann-Hilbert integral equation, occurring in the classical problem of transport phenomena in a rigid-sphere gas model, into a differential equation. In the case of self-diffusion treated there, this differential equation was of the second order, and its solution yielded a value for the coefficient of self-diffusion which was in good agreement with the value obtained by the variational Chapman-Enskog method. In this paper the method is applied to the problems of heat conduction and viscosity. In both cases the differential equations for the respective distribution functions are of the fourth order. The solution of these equations leads to values for the coefficients of heat conduction and of viscosity which are in good agreement with the values obtained by the Chapman-Enskog method. From the tabulated values of the distribution functions it follows that the linearized approximation becomes relatively poorer in the outer regions of momentum-space. A differential equation of the fourth order for the distribution function in the case of viscosity was derived by L. Boltzmann.³ Boltzmann's differential equation is incorrect as it stands because of errors that crept into the last stages of his derivation. Boltzmann did not integrate the differential equation. To the authors' knowledge the differential equation governing the distribution function in the case of heat conduction, which is derived and solved in this paper, is new.

¹ C. L. Pekeris, these PROCEEDINGS, 41, 661, 1955. This paper will be referred to as "Paper I."

² *Ibid.*, eq. (18).

³ L. Boltzmann, *Collected Works*, 2 (1881), 545.

⁴ *Ibid.*, eq. (37), p. 544. Here $\psi(p^2) = 2b(p)$.

⁵ *Ibid.*, pp. 479-522.

⁶ See S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases* (Cambridge: At the University Press, 1952), pp. 109 and 121.

⁷ *Ibid.*, p. 169.

PROTEIN SYNTHESIS AND TISSUE INTEGRITY IN THE CORNEA OF THE DEVELOPING CHICK EMBRYO*

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In the course of investigations of protein formation in the tissues of the chick embryo, two systems have been used so far in this laboratory for the comparison of tracer incorporation and protein accumulation in developing cells: isolated protein fractions of muscle tissue obtained from embryos developing *in ovo*¹ and the total protein moiety of early explanted embryos.^{2, 3} The first approach offers advantages for the preparation of well-defined protein fractions. But the interpretation of the data obtained from *in vivo* systems is complicated by the changing relationships of the tracer levels in the blood, in the cell pool, and in the proteins of the embryo.